$$\kappa_i = -\frac{1}{2(l_1 + l_2)} \ln (P^2 + Q^2). \tag{31}$$

This problem has been solved numerically (e.g., Fig. 1), primarily in an attempt to see whether, for an electron-hole plasma system, the structure provides an additional space-harmonic coupling mechanism giving rise to amplification, i.e., a change in the sign of κ_i. We conclude that no gain can be directly attributed to the structure, although gain is evident when at least one of the constituent basis functions (infinite medium solutions) exhibits amplification. Coupling to space harmonics evidently requires off-axis propagation, since only in this manner can a longitudinal ac electric field be provided.

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Variational Expressions, Pseudoenergy, and Power

P. DALY

***Abstract*—A variational expression involving the longitudinal fields in an inhomogeneous lossless waveguide, which acts as the starting point for numerical procedures, is shown to be equivalent to a simple relationship between power flow, stored energy, and phase velocity.**

Recently Chorney [1] and Chorney and Penfield [2] have developed several interesting results in various uniform waveguide systems concerning power flow and pseudoenergy. A particular result which has been derived for a lossless passive uniform waveguide relates the power flow P to the pseudoenergies U_T and U_z:

$$P - \frac{\omega}{\beta}(U_T - U_z) = 0 \tag{1}$$

where U_T and U_z are the energies stored per unit length in the transverse and longitudinal fields, respectively, and ω/β is the phase velocity. What is even more interesting about the above expression is that, at least for nondispersive isotropic materials it is a variational expression in the unknown waveguide fields and represents the starting point for several numerical solutions of the waveguide problem including the finite-element method.

Consider an inhomogeneous lossless uniform waveguide with field dependence $\exp[j(\omega t - \beta z)]$ whose field components may be defined [3], [4] in terms of two scalars ϕ and ψ as

$$H_z = \phi \tag{2}$$

$$E_z = \frac{\beta\psi}{\omega\epsilon_0} \tag{3}$$

$$k^2 \boldsymbol{H}_t = -j\beta[\nabla_t\phi + \kappa(\boldsymbol{e}_z \times \nabla_t\psi)] \tag{4}$$

$$k^2 \boldsymbol{E}_t = j\omega\mu_0[\mu(\boldsymbol{e}_z \times \nabla_t\phi) - \bar{\beta}^2\nabla_t\psi] \tag{5}$$

where κ and μ are relative dielectric constants and permeabilities, respectively.

$$k^2 = \left(\frac{\omega}{c}\right)^2 \kappa\mu - \beta^2 \qquad \bar{\beta} = \frac{\beta c}{\omega}. \tag{6}$$

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The total power flow over the waveguide cross section is

$$\begin{aligned} P &= \frac{1}{2}\int\int \boldsymbol{e}_z\cdot(\boldsymbol{E}_t \times \boldsymbol{H}_t^*)\, dS \\ &= \frac{\omega\mu_0\beta}{2}\int\int \frac{1}{k^4}[\mu|\nabla_t\phi|^2 + \kappa\bar{\beta}^2|\nabla_t\psi|^2 + (\kappa\mu + \bar{\beta}^2)\boldsymbol{e}_z \\ &\quad \cdot(\nabla_t\psi \times \nabla_t\phi)]\, dS. \end{aligned} \tag{7}$$

Defining the terms involving stored energy per unit length, we have longitudinal pseudoenergy:

$$\begin{aligned} U_z &= \frac{1}{4}\int\int (H_z^* B_z + E_z D_z^*)\, dS \\ &= \frac{\mu_0}{4}\int\int [\mu\phi^2 + \kappa\bar{\beta}^2\psi^2]\, dS \end{aligned} \tag{8}$$

transverse magnetic pseudoenergy:

$$\begin{aligned} U_{mT} &= \frac{1}{4}\int\int \boldsymbol{H}_t^*\cdot\boldsymbol{B}_t\, dS \\ &= \frac{\mu_0\beta^2}{4}\int\int_S \frac{\mu}{k^4}[|\nabla_t\phi|^2 + 2\kappa\boldsymbol{e}_z\cdot(\nabla_t\psi \times \nabla_t\phi) + \kappa^2|\nabla_t\psi|^2]\, dS \end{aligned} \tag{9}$$

transverse electric pseudoenergy:

$$\begin{aligned} U_{eT} &= \frac{1}{4}\int\int_S \boldsymbol{E}_t\cdot\boldsymbol{D}_t^*\, dS \\ &= \frac{\mu_0}{4}\left(\frac{\omega}{c}\right)^2\int\int_S \frac{\kappa}{k^4}[\mu^2|\nabla_t\phi|^2 - 2\mu\bar{\beta}^2\boldsymbol{e}_z\cdot(\nabla_t\phi \times \nabla_t\psi) \\ &\quad + \bar{\beta}^4|\nabla_t\psi|^2]\, dS \end{aligned} \tag{10}$$

transverse pseudoenergy:

$$\begin{aligned} U_T &= U_{mT} + U_{eT} \\ &= \frac{\mu_0}{4}\left(\frac{\omega}{c}\right)^2\int\int_S \frac{1}{k^4}[\mu(\kappa\mu + \bar{\beta}^2)|\nabla_t\phi|^2 + \kappa\bar{\beta}^2(\kappa\mu + \bar{\beta}^2)|\nabla_t\psi|^2 \\ &\quad + 4\mu\kappa\bar{\beta}^2\boldsymbol{e}_z\cdot(\nabla_t\psi \times \nabla_t\phi)]\, dS. \end{aligned} \tag{11}$$

Substituting for P, U_z, and U_T in (1) and rearranging, we obtain

$$\left(\frac{\omega}{c}\right)^2\int\int_S [\mu\phi^2 + \kappa\bar{\beta}^2\psi^2]\, dS - \int\int_S \frac{\mu|\nabla_t\phi|^2 + \kappa\bar{\beta}^2|\nabla_t\psi|^2 + 2\bar{\beta}^2\boldsymbol{e}_z\cdot(\nabla_t\psi \times \nabla_t\phi)}{\kappa\mu - \bar{\beta}^2}\, dS = 0. \tag{12}$$

Expression (12) is, however, a variational expression in the scalar functions ϕ and ψ which was derived [4], [5] in connection with a finite-element analysis of hybrid-modes in waveguide. The result in [4] and [5] was derived for $\mu = 1$ but the generalization to (12) is simple.

As a further point of interest, it is worth noting that variational expressions are often written directly in terms of vector fields. Such an expression is to be found in the work of English [6], who derives the following vector variational expression [7]:

$$\begin{aligned} &\beta\int\int_S [\boldsymbol{H}^*\cdot(\boldsymbol{e}_z \times \boldsymbol{E}) - \boldsymbol{E}^*\cdot(\boldsymbol{e}_z \times \boldsymbol{H})]\, dS \\ &\quad - \omega\int\int_S (\boldsymbol{E}^*\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{E} + \boldsymbol{H}^*\cdot\boldsymbol{\mu}\cdot\boldsymbol{H})\, dS \\ &\quad - j\int\int_S [\boldsymbol{E}^*\cdot(\nabla_t \times \boldsymbol{H}) - \boldsymbol{H}^*\cdot(\nabla_t \times \boldsymbol{E})]\, dS = 0 \end{aligned} \tag{13}$$

subject to the boundary condition $\boldsymbol{n}\times\boldsymbol{E} = 0$.

Under the condition that the medium is isotropic and nondispersive, (13) can easily be shown by direct substitution from (2)–(5) to be another reformulation of (1). It is interesting to speculate that perhaps (1) remains a variational expression under conditions more stringent than those imposed here, e.g., when the medium is dispersive or anisotropic.

Finally, it is shown by Chorney that for propagating modes, total electric and magnetic stored energies are equal. Using the subscripts m and e to denote magnetic and electric, respectively, we obtain

$$U_m - U_e = \frac{\mu_0}{4}\int\int_S [\mu\phi^2 - \kappa\bar{\beta}^2\psi^2]\, dS - \frac{\mu_0}{4}\int\int_S \frac{1}{k^2}[\mu|\nabla_t\phi|^2 - \kappa\bar{\beta}^2|\nabla_t\psi|^2]\, dS = 0. \quad (14)$$

In expression (14), the term coupling ϕ and ψ together on the boundary disappears but the expression is not variational. This is because the integrand is not necessarily positive definite.

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Q-Dependence of Gunn Oscillator FM Noise

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***Abstract*—Measurements of FM noise and the external quality factor Q_{ex} of X-band Gunn oscillators are reported which show that both upconverted and intrinsic FM noise vary inversely as Q_{ex}, if bias voltage, RF power, and frequency are kept constant.**

The FM noise spectrum of a Gunn oscillator consists of two parts, the intrinsic noise and the upconverted noise. In this letter it is shown experimentally that both parts depend inversely on the quality factor Q_{ex} of the resonant circuit. This result is in agreement with theory.

At low modulation frequencies f_m the FM noise of a Gunn oscillator is ruled by upconversion of low-frequency fluctuations of the device impedance. The mean-square noise current usually decreases as $1/f_m$ resulting in a $1/f_m^{1/2}$ behavior of the mean frequency deviation Δf_{rms}. Towards higher frequencies off the carrier (above 100 kHz) white intrinsic noise due to amplification of thermal noise predominates.

Recently, successful efforts have been made to reduce FM noise by a stabilizing high-Q cavity [1], whereas AM noise remained essentially unchanged within the bandwidth of the cavity [2]. For this reason and because AM noise power is much less than FM noise power for Gunn oscillators near the carrier, only FM noise is treated in this work. For practical applications, it is of considerable interest to know the Q_{ex} dependence of both parts of FM noise. Several authors have investigated this problem. Cathelin *et al.* [3] measured $\Delta f_{rms}\sqrt{Q_{ex}}$ to be fairly independent of Q_{ex} for modulation frequencies between 2 and 100 kHz. Sweet and MacKenzie [4] found Δf_{rms} to depend on the voltage pushing only and not on Q_{ex} for upconverted noise and to vary as $(Q_{ex}\cdot\sqrt{P_0})^{-1}$ for intrinsic noise, where P_0 is the RF power of the oscillator. The contradiction in these results is mainly because of the fact that the investigations did not take into account the dependence of the noise on the device impedance and its derivatives with respect to the modulating parameters or the RF voltage amplitude.

The basic equation for both upconverted and intrinsic noise is the condition of oscillation [5]:

$$A(t)[Y_D(M(t)) + Y_L(f)] + e_c(t) + je_s(t) = 0. \quad (1)$$

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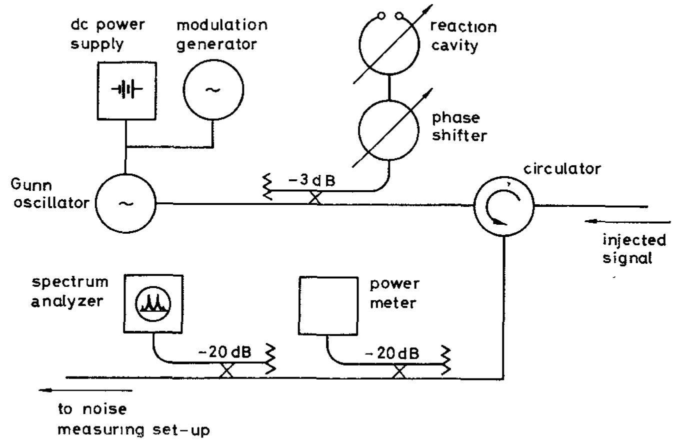

Fig. 1. Schematical plot of the Q-measuring setup.

$Y_D = G_D + jB_D$ and $Y_L = G_L + jB_L$ are the admittances of the device and the load, respectively. $A(t)$, $M(t)$, $e_c(t)$, and $e_s(t)$ are slowly varying functions of time. $A(t)$ is the RF voltage amplitude, $M(t)$ some modulating parameter as, e.g., carrier density, bias voltage, or temperature, giving rise to upconverted noise, and $e_c(t)$ and $e_s(t)$ represent the primary noise source near the carrier frequency f_0. The device admittance may only depend on M and not on the frequency f, which is an appropriate description for a Gunn element. Small fluctuations of the modulating parameter M result in fluctuations of the diode admittance Y_D which have to be taken into account in (1) as

$$\Delta Y_D(t) = \frac{\partial Y_D}{\partial M}\Delta M(t) = \frac{1}{A}(a_c(t) + ja_s(t)). \quad (2)$$

Neglecting all terms causing AM noise, both upconverted and intrinsic FM noise are described by a single expression as pointed out by Kurokawa [5]:

$$\Delta f_{rms} = \frac{f_0}{Q_{ex}}\sqrt{\frac{(e_s)_{rms}{}^2 + (a_s(f))_{rms}{}^2}{4|G_D|P_0}} \quad (3)$$

$$= \frac{f_0}{Q_{ex}}\sqrt{\frac{kT_{eq}(f)B}{P_0}} \quad (4)$$

where k is Boltzmann's constant, T_{eq} the frequency dependent equivalent noise temperature of the noise sources involved, and B is the measuring bandwidth. Equation (4) was first derived by Edson [6] for the case of pure intrinsic noise.

For upconverted noise it is obvious from (2) and (3) that Δf_{rms} varies inversely as Q_{ex} if carrier frequency f_0, output power P_0, and bias voltage V_B are kept constant. The same holds for intrinsic noise because the primary RF noise source depends on the negative conductance [7] and, consequently, on the output power P_0 of the Gunn oscillator. Therefore, the relation $\Delta f_{rms}\sqrt{P_0}\sim Q_{ex}^{-1}$, often taken as evidence for intrinsic noise, only holds for the special condition that T_{eq} is independent of P_0. This can only be the case in a small range of operation parameters.

Equation (4) does not take into account AM–FM and FM–AM conversion. This is done by considering the dependence of the device admittance on the RF amplitude A. A linearized theory [8] leads to an expression for the frequency fluctuation:

$$\Delta f(t) = \frac{f_0\left[-i_s(t) + i_c(t)\left(\frac{\partial B_D}{\partial A}\Big/\frac{\partial G_D}{\partial A}\right)\right]}{2AG_L\cdot Q_{ex}\left[1 - \left(\frac{\partial B_D}{\partial A}\Big/\frac{\partial G_D}{\partial A}\right)\left(\frac{\partial G_L}{\partial f}\Big/\frac{\partial B_L}{\partial f}\right)\right]} \quad (5)$$

where $i_s(t) = e_s(t) + a_s(t)$ and $i_c(t) = e_c(t) + a_c(t)$. AM–FM conversion is described by the second term of the numerator, whereas the second term in the brackets of the denominator represents FM–AM conversion. With $Q_{ex} = (f_0/2G_L)(\partial B_L/\partial f)$, one obtains that Δf_{rms} will always vary inversely as Q_{ex} when FM–AM conversion can be neglected.

Q_{ex} measurements were carried out using the modulation method of Ashley and Palka [9]. The measuring setup is shown in Fig. 1. This method allows to determine the quality factor rather exactly